INVERSE SCATTERING RESULTS FOR MANIFOLDS HYPERBOLIC NEAR INFINITY

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Abstract. We study the inverse resonance problem for conformally compact manifolds which are hyperbolic outside a compact set. Our results include compactness of isoresonant metrics in dimension two and of isophasal negatively curved metrics in dimension three. In dimensions four or higher we prove topological finiteness theorems under the negative curvature assumption.

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1. Introduction

The inverse problem of recovering an asymptotically hyperbolic metric from the associated scattering data has many possible variants, depending on how much knowledge is assumed. It is well-known that the resonance set does not determine an asymptotically hyperbolic manifold completely, even in the exactly hyperbolic case. See, for example, Guillopé-Zworski [29, Remark 2.15], Brooks–Gornet–Perry [12], Brooks-Davidovitch [11], and the survey paper Gordon-Perry-Schueth [20]. One can however obtain strong positive results by assuming knowledge of the scattering matrix itself. For surfaces, a result of Lassas-Uhlmann [35] shows that the scattering matrix at the point $s = 1$ determines the metric up to isometry. The corresponding result for even dimensional conformally compact Einstein manifolds was proven by Guillarmou-Sá Barreto [27]. Another recent inverse result of Sá Barreto [43] shows that an asymptotically hyperbolic manifold is completely determined by scattering matrix at all energies. Note that one must fix the boundary at infinity to make sense of the assumption that two scattering matrices are equal.

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In between these two extremes, another standard assumption in scattering theory is that the metrics are *isophasal*, meaning that they share the same *scattering phase*. Defining the scattering phase requires some regularization of scattering determinants. In the even dimensional asymptotically hyperbolic case, Guillarmou [25] shows that a canonical regularization can be defined. In the odd dimensional case, we can only define a relative scattering phase between two manifolds that are isometric near infinity. For all of the isoscattering examples cited above, the resonance sets can be identified because the respective scattering matrices are intertwined by transplantation operators. For hyperbolic surfaces the transplantation method gives examples that are isophasal, as noted in [29, Remark 2.15]. The three-dimensional isoscattering pairs are not necessarily isometric near infinity, so it's not even clear that the relative scattering phase is well-defined in these cases.

For conformally compact manifolds which are hyperbolic near infinity (i.e. outside a compact set), the Hadamard factorization of the relative scattering determinant from Borthwick [7, Prop. 7.2] shows that the resonance set determines the scattering phase (relative to some fixed background metric) up to a polynomial of degree $n+1$. Thus, assuming a background metric is fixed, the isophasal condition is only slightly stronger than isoresonance, in the sense that it requires the equality of only a few additional parameters.

The purpose of this note is to prove topological finiteness and geometric compactness results in the context of conformally compact manifolds hyperbolic near infinity, for isoresonant classes in even dimensions and isophasal classes in odd dimensions.

For (X, g) conformally compact and hyperbolic near infinity, we let $\dim X = n+1$ and denote by Δ_g the positive Laplacian associated to g. The resolvent $R_g(s) := (\Delta_g - s(n-s))^{-1}$ has a meromorphic continuation to $s \in \mathbb{C}$ with poles of finite rank [36, 28]. The *resonance set* $\mathcal{R}_g$ is the set of poles of $R_g(s)$, counted according to the multiplicity given by

$$m_g(\zeta) := \operatorname{rank} \operatorname{Res}_\zeta R_g(s).$$

Resonances are closely related to the poles of the scattering matrix $S_g(s)$, defined as in [32, 21]. Let ρ be a boundary defining function for the conformal compactification $\bar{X}$. For $\operatorname{Re} s = \frac{n}{2}$, $s \neq \frac{n}{2}$, a function $f_1 \in C^\infty(\partial_\infty X)$ determines a unique solution of $(\Delta_g - s(n-s))u = 0$ such that

$$u \sim \rho^{n-s} f_1 + \rho^s f_2$$

as $\rho \to 0$, with $f_2 \in C^\infty(\partial_\infty X)$ This defines the map $S_g(s) : f_1 \mapsto f_2$, which extends meromorphically to $s \in \mathbb{C}$ as a family of pseudodifferential operators of order $2s - n$. To define a scattering determinant, we will fix a background metric g_0 and use S_{g_0} as a reference operator. If metrics g, g_0 agree to $O(\rho^\infty)$, then the product $S_g(s) S_{g_0}(s) - I$ is smoothing [32] and so the relative scattering determinant,

$$\tau(s) := \det S_g(s) S_{g_0}(s)^{-1}, \tag{1.1}$$

is well-defined as a Fredholm determinant. When restricted to the critical line $\operatorname{Re} s = \frac{n}{2}$, we have $|\tau(s)| = 1$, and the relative scattering phase is a real-valued function (for real ξ) defined by

$$\sigma(\xi) := \frac{i}{2\pi} \log \tau(\tfrac{n}{2} + i\xi), \tag{1.2}$$

with branches chosen so that $\sigma(\xi)$ is continuous starting from $\sigma(0) = 0$. By the symmetry properties of the scattering matrix, $\sigma(-\xi) = -\sigma(\xi)$. The scattering matrices depend on the choice of ρ, but $\tau(s)$ and $\sigma(\xi)$ are invariantly defined.

To state our results, fix a conformally compact manifold (X_0, g_0) of dimension $n+1$ with a compact subset $K_0 \subset X_0$ such that g_0 is hyperbolic outside K_0 (meaning sectional curvatures $= -1$). We wish to allow arbitrary metric perturbations within K_0, and so consider the class

$$\mathcal{M}(X_0, g_0, K_0) := \Big\{(X, g) : \ (X - K, g) \cong (X_0 - K_0, g_0) \text{ for some } K \subset X\Big\}, \tag{1.3}$$

where $\cong$ denotes Riemannian isometry. For each X_0, g_0 we will fix a boundary defining function ρ and then use this same function for the entire class $\mathcal{M}(X_0, g_0, K_0)$.

Naturally, the strongest results are possible in the case of surfaces:

Theorem 1.1. *Fix X_0, g_0, K_0 as above with $\dim X_0 = 2$. If $\mathcal{A} \subset \mathcal{M}(X_0, g_0, K_0)$ is a collection of surfaces (X, g) that share a common resonance set $\mathcal{R}$, then $\mathcal{A}$ is compact in the C^∞ topology.*

This of course is analogous to the well-known result of Osgood-Phillips-Sarnak [38] for compact surfaces. And it is a considerable improvement over the comparable result of Borthwick-Judge-Perry [8, Thm 1.4], for which the metric perturbations were restricted to conformal deformations with compactly supported conformal parameter. (See §7 for some explanation of the improvement.)

In three dimensions we require more restrictive geometric assumptions and more scattering data to produce a comparable result:

Theorem 1.2. *Fix (X_0, g_0) and $K_0 \subset X_0$ as above with $\dim X_0 = 3$. Assume that $\mathcal{A} \subset \mathcal{M}(X_0, g_0, K_0)$ is a set of 3-manifolds (X, g) with negative sectional curvatures which share a common scattering phase. Then $\mathcal{A}$ is compact in the C^∞ topology.*

Note that the isophasal condition could be expressed without reference to the scattering matrix of (X_0, g_0) by requiring that the relative scattering phase between any pair of manifolds in $\mathcal{A}$ is zero. In practice it will be more convenient to define relative phases $\sigma_g(\xi)$ with respect to the fixed background g_0.

Theorem 1.2 is closely analogous to compactness results obtained for isospectral compact 3-manifolds by Anderson [4] and Brooks-Perry-Petersen [14]. In dimensions greater than three, the conclusions are limited to topological finiteness, just as in the corresponding results of [14].

Theorem 1.3. *Fix X_0, g_0, K_0 as above with $\dim X_0 = n + 1 \geq 4$. Assume that $\mathcal{A} \subset \mathcal{M}(X_0, g_0, K_0)$ is a set of $(n + 1)$-manifolds (X, g) with negative sectional curvatures which share either*

- *a common resonance set $\mathcal{R}$ if $\dim X$ is even, or*
- *a common scattering phase $\sigma(\xi)$ if $\dim X$ is odd.*

Then $\mathcal{A}$ contains only finitely many homeomorphism types, and for $\dim X > 4$ at most finitely many diffeomorphism types.

The paper is organized as follows. In §2–4 we review the scattering theory and the various results that allow one to deduce geometric information from it. The proof of Theorem 1.3 is given in §5. In §6 we review some geometric compactness results and apply these to give the proofs of Theorems 1.1 and 1.2. The proof for surfaces is the most complicated, in that we must establish curvature bounds

without any control of the injectivity radius at the outset. This part of the proof, which is based on conformal uniformization, is deferred to §7.

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2. Poisson formula

Resonances are closely related to the poles of the scattering matrix $S_g(s)$, defined as in [32, 21]. This operator has infinite-rank poles, so to define multiplicities of scattering poles, we use a renormalized scattering matrix of order zero given by

$$\tilde{S}_g(s) := \frac{\Gamma(s-\frac{n}{2})}{\Gamma(\frac{n}{2}-s)} \Lambda^{n/2-s} S_g(s) \Lambda^{n/2-s}. \tag{2.1}$$

where

$$\Lambda := \frac{1}{2}(\Delta_h + 1)^{1/2}.$$

This renormalization makes $\tilde{S}_g(s)$ into a meromorphic family of Fredholm operators with poles of finite rank. The multiplicity at a pole or zero of $S_g(s)$ is then defined by

$$\nu_g(\zeta) := -\operatorname{tr}\big[\operatorname{Res}_\zeta \tilde{S}_g'(s) \tilde{S}_g(s)^{-1}\big]$$

(with poles counted positively to match the resonances). The dependence of $\tilde{S}_g(s)$ on the boundary defining function ρ is wiped out by the trace, so that $\nu_g(\zeta)$ is invariantly defined.

The scattering multiplicities are related to the resonance multiplicities by results of Guillopé-Zworksi [29], Borthwick-Perry [9] and Guillarmou [24] (with a restriction that was later removed in [26]):

$$\nu_g(\zeta) = m_g(\zeta) - m_g(n-\zeta) + \sum_{k\in\mathbb{N}} \Big(\mathbb{1}_{n/2-k}(\zeta) - \mathbb{1}_{n/2+k}(\zeta) \Big) d_k, \tag{2.2}$$

where $\mathbb{1}_p$ denotes the characteristic function on $\{p\}$ and

$$d_k := \dim\ker \tilde{S}_g(\tfrac{n}{2}+k).$$

From Graham-Zworski [21] it follows that the d_k's are invariants of the conformal structure induced on $\partial_\infty X$ by the metric $\rho^2 g$. For surfaces ($n = 1$), the d_k terms always vanish [6, Lemma 8.6]. But in higher dimensions they may occur and even saturate the resonance counting function (see [26] or [7]).

To state certain results, such as the Poisson formula, we need to incorporate these extra scattering poles into a *scattering resonance* set,

$$\mathcal{R}_g^{\mathrm{sc}} := \mathcal{R}_g \cup \bigcup_{k=1}^{\infty} \Big\{ \tfrac{n}{2} - k \text{ with multiplicity } d_k \Big\}.$$

For any inverse scattering problem, it makes sense to assume that the d_k's are fixed, since they depend only on the structure at infinity.

We will state inverse scattering results in two different contexts. First, we showing that certain geometric information that can be deduced solely from $\mathcal{R}_g^{\mathrm{sc}}$, without assuming knowledge of (X_0, g_0). The catch is that for this purpose we must assume that (X_0, g_0) is exactly hyperbolic. Later in the section, we'll give inverse results that apply within $\mathcal{M}(X_0, g_0, K_0)$. This is the context of §1, for which we assume

knowledge of (X_0, g_0) and $\mathcal{R}_{g_0}$, but drop the assumption that the background is exactly hyperbolic.

In the case of a compactly supported perturbation of a conformally compact hyperbolic metric, Borthwick [7] gave a Poisson formula for resonances that relates the regularized wave trace, defined as a distribution on $\mathbb{R}$ by

$$\Theta_g(t) := 0\text{-tr}\left[\cos\left(t\sqrt{\Delta_g - n^2/4}\right)\right],$$

to a sum over $\mathcal{R}_g^{\mathrm{sc}}$. The assumption the the background is exactly hyperbolic allows contributions from the background metric to be cancelled from both sides of a relative Poisson formula, yielding a result that has no explicit dependence on (X_0, g_0) or $\mathcal{R}_{g_0}^{\mathrm{sc}}$.

Theorem 2.1 (Poisson formula). *Let (X, g) be a compactly supported perturbation of a conformally compact hyperbolic manifold. Then, in a distributional sense on $\mathbb{R} - \{0\}$,*

$$\Theta_g(t) = \frac{1}{2}\sum_{\zeta\in\mathcal{R}_g^{\mathrm{sc}}} e^{(\zeta-n/2)|t|} - A(X)\frac{\cosh t/2}{(2\sinh|t|/2)^{n+1}},$$

where

$$A(X) := \begin{cases} 0 & n \text{ odd } (\dim X \text{ is even}), \\ \chi(X) & n \text{ even } (\dim X \text{ is odd}). \end{cases}$$

Note that in odd dimensions we could also write $A(X)$ as $\frac{1}{2}\chi(\partial_\infty X)$.

In two dimensions this formula is due to Guillopé and Zworski [30], and the requirement for an exactly hyperbolic background metric is not necessary for that case. For hyperbolic manifolds of any dimension it was proved by Guillarmou and Naud [26]. The result as stated here is Borthwick [7, Thm. 1.2]

Corollary 2.2. *Assume (X, g) is a compactly supported perturbation of a conformally compact hyperbolic manifold. In the even-dimensional case (n odd), the set $\mathcal{R}_g^{\mathrm{sc}}$ determines the wave 0-trace as a distribution on $\mathbb{R}$, and fixes* 0-vol(X, g) *in particular. In odd dimensions (n even), $\mathcal{R}_g^{\mathrm{sc}}$ determines $\chi(X)$ and the restriction of the wave trace to $t \neq 0$.*

Proof. Joshi and Sá Barreto [33] showed that the asymptotic expansion of the wave 0-trace at $t = 0$ has the same form as found by Duistermaat-Guillemin [19]. That is, if $\psi \in C_0^\infty(\mathbb{R})$ has support in a sufficiently small neighborhood of 0 and $\psi = 1$ in some smaller neighborhood of 0, then

$$\int_{-\infty}^{\infty} e^{-it\xi}\psi(t)\Theta_g(t)\,dt \sim \sum_{k=0}^{\infty} a_k|\xi|^{n-2k}, \tag{2.3}$$

where

$$a_0 = \frac{2^{-n}\pi^{-\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}\,0\text{-vol}(X, g).$$

In even dimensions, the powers $|\xi|^{n-2k}$ correspond to singularities of the form $t^{-n-1+2k}$ (homogeneous regularization). Thus the singularities are detectable in the behavior of the wave 0-trace as $t \to 0_+$. By the Poisson formula, $\mathcal{R}_g^{\mathrm{sc}}$ determines the wave trace completely for $t \neq 0$, and so the wave coefficients $\{a_k\}$ are also fixed by $\mathcal{R}_g^{\mathrm{sc}}$.

In the odd dimensional case (n even), $|\xi|^{n-2k}$ corresponds to $\delta^{(n-2k)}(t)$ when $n-2k \geq 0$. Thus, the singularity of the wave 0-trace at $t=0$ is not computable from $\mathcal{R}_g^{\mathrm{sc}}$. (Indeed, in odd dimensions a_0 depends on the choice of boundary defining function ρ, so to obtain a_0 from $\mathcal{R}_g^{\mathrm{sc}}$ is impossible a priori.) Since the wave-trace singularities are localized at $t=0$, one sees only the blowup caused by the $\chi(X)$ term as $t \to 0_+$. Hence $\chi(X)$ is fixed by $\mathcal{R}_g^{\mathrm{sc}}$. □

Joshi and Sá Barreto [33] also showed that the wave 0-trace for an asymptotically hyperbolic manifold has singularities for $t \neq 0$ contained in the set of lengths of closed geodesics of X. In the case when the sectional curvatures of (X,g) are strictly negative, Rowlett [42, Thm 1.1] has recently refined this result to show that, for $t \geq \varepsilon > 0$ we have

$$\Theta_g(t) = \sum_{\ell \in \mathcal{L}_g} \sum_{k=1}^{\infty} \frac{\ell}{\sqrt{|\det 1 - P_\ell^k|}} \delta(t - k\ell) + R(t), \tag{2.4}$$

where $\mathcal{L}_g$ is the primitive length spectrum of (X,g), P_ℓ^k is the k-times around Poincaré map for the geodesic associated to ℓ, and the remainder $R(t)$ is smooth and bounded on $[\varepsilon, \infty)$. This immediately leads the following:

Corollary 2.3. *Assuming that (X,g) is a compactly supported perturbation of a conformally compact hyperbolic manifold with strictly negative sectional curvatures, the resonance set $\mathcal{R}_g^{\mathrm{sc}}$ determines the length spectrum of (X,g), and in particular fixes the injectivity radius* $\mathrm{inj}(X,g)$.

(Note that under the negative curvature assumption, $\mathrm{inj}(X,g)$ is equal to half the length of the shortest closed geodesic.)

We now turn to the results needed for the applications given in §1, for which we can assume full knowledge of the fixed background (X_0, g_0). In this situation, we can start from a relative Poisson formula, which does not require the background to be exactly hyperbolic.

Theorem 2.4. *For $(X,g) \subset \mathcal{M}(X_0, g_0, K_0)$ defined as in (1.3), where (X_0, g_0) is conformally compact and hyperbolic near infinity, we have*

$$\Theta_g(t) - \Theta_{g_0}(t) = \frac{1}{2} \sum_{\zeta \in \mathcal{R}_g} e^{(\zeta - n/2)|t|} - \frac{1}{2} \sum_{\zeta \in \mathcal{R}_{g_0}} e^{(\zeta - n/2)|t|}.$$

Proof. We define the meromorphic function,

$$\Upsilon_g(s) := (2s-n)\, \text{0-tr}[R_g(s) - R_g(n-s)],$$

for $s \notin \mathbb{Z}/2$. By [7, Lemma 7.1 and Prop. 7.2], we have

$$\Upsilon_g(s) - \Upsilon_{g_0}(s) = \partial_s \log \left[e^{q(s)} \frac{P_g(n-s)}{P_g(s)} \frac{P_{g_0}(s)}{P_{g_0}(n-s)} \right], \tag{2.5}$$

where $P_*(s)$ denotes the Hadamard product over the resonance set $\mathcal{R}_*$, and $q(s)$ is a polynomial. (We can use $\mathcal{R}_*$ rather than $\mathcal{R}_*^{\mathrm{sc}}$ here, because the extra d_k terms are canceled by the background.) On the other hand, [7, Eq. (8.2) and Lemma 8.1] show that $\Upsilon_g(s)$ is essentially the inverse Fourier transform of the continuous part of the wave trace. Taking the Fourier transform of (2.5), exactly as in the proof of [7, Thm. 1.2], yields the formula given above. □

Corollary 2.5. *Assuming* $\dim X_0$ *is even, for metrics in* $\mathcal{M}(X_0, g_0, K_0)$ *the resonance set* $\mathcal{R}_g$ *determines* $vol(K, g)$*. In any dimension, for metrics of strictly negative sectional curvatures in* $\mathcal{M}(X_0, g_0, K_0)$*, the resonance set* $\mathcal{R}_g$ *determines* $\mathrm{inj}(X, g)$.

Proof. Using (2.3), in the even dimensional case we can deduce

$$0\text{-vol}(X, g) - 0\text{-vol}(X_0, g_0) = \mathrm{vol}(K, g) - \mathrm{vol}(K_0, g_0)$$

from $\mathcal{R}_g$ and $\mathcal{R}_{g_0}$. Hence, within $\mathcal{M}(X_0, g_0, K_0)$ we see that $\mathcal{R}_g$ determines $\mathrm{vol}(K, g)$. Similarly, since $\Theta_{g_0}(t)$ is fixed within $\mathcal{M}(X_0, g_0, K_0)$, from (2.4) we see that $\mathcal{R}_g$ determines the length spectrum for metrics of negative sectional curvature, and hence the injectivity radius. □

3. Relative scattering phase

For $X \in \mathcal{M}(X_0, K_0, g_0)$, the relative scattering determinant $\tau(s)$ and scattering phase $\sigma(\xi)$ were defined in (1.1) and (1.2), respectively. Since $\tau(s)$ is meromorphic, fixing $\sigma(\xi)$ determines $\tau(s)$ as well. Define the Weierstrass product

$$P_g(s) := \prod_{\zeta \in \mathcal{R}_g^{\mathrm{sc}}} E\Big(\frac{s}{\zeta}, n+1\Big), \tag{3.1}$$

where $E(w, k)$ is an elementary factor,

$$E(w, k) := (1 - w)e^{w + w^2/2 + \cdots + w^k/k}.$$

Let $P_{g_0}(s)$ be the corresponding product for $\mathcal{R}_{g_0}^{\mathrm{sc}}$. By [7, Prop. 7.2],

$$\tau(s) = e^{q(s)} \frac{P_g(n - s)}{P_g(s)} \frac{P_{g_0}(s)}{P_{g_0}(n - s)}, \tag{3.2}$$

where $q(s)$ is a polynomial of degree at most $n+1$. The coefficients of $q(s)$, which has the symmetry $q(s) = -q(n - s)$, are the extra parameters that we fix by assuming equality of scattering phases instead of resonance sets. In the other direction, the factorization formula (3.2) makes it clear that $\sigma(\xi)$ determines $\mathcal{R}_g^{\mathrm{sc}}$, modulo the fixed background $\mathcal{R}_{g_0}^{\mathrm{sc}}$.

Another important formula for the relative scattering phase connects it to the (regularized) traces of the spectral resolutions. For $s \neq \mathbb{Z}/2$ we have

$$\frac{\partial \sigma}{\partial \xi}(\xi) = \frac{i\xi}{\pi}\Big(0\text{-tr}\big[R_g(\tfrac{n}{2} + i\xi) - R_g(\tfrac{n}{2} - i\xi)\big] - 0\text{-tr}\big[R_{g_0}(\tfrac{n}{2} + i\xi) - R_{g_0}(\tfrac{n}{2} - i\xi)\big]\Big).$$

By the functional calculus, the two terms on the right are the Fourier transforms of the continuous parts of the respective regularized wave traces, except at $\xi = 0$, where the 0-trace can have an anomaly. By [7, (8.1–2)], we deduce the following:

Proposition 3.1. *For* (X_0, g_0) *conformally compact and hyperbolic near infinity and* $(X, g) \in \mathcal{M}(X_0, g_0, K_0)$*, the relative scattering phase* $\sigma(\xi)$ *determines the relative wave trace* $\Theta_g(t) - \Theta_{g_0}(t)$*, as a distribution for* $t \in \mathbb{R}$.

Note that the big singularity of the wave trace at $t = 0$ is included in this result, because it corresponds to the behavior of $\sigma(\xi)$ as $|\xi| \to \infty$.

4. Relative heat invariants

Suppose that (X, g) is conformally compact and hyperbolic near infinity, and let $H_g(t; z, z')$ denote the heat kernel associated to Δ_g. The restriction of the heat kernel to the diagonal has the usual local expansion as $t \to 0$,

$$H_g(t; z, z) \sim t^{-\frac{n+1}{2}} \sum_{j=0}^{\infty} t^j \alpha_j(g; z). \tag{4.1}$$

In our setting, the heat operator is not trace class, and the local geometric invariants $\alpha_j(g)$ are not integrable over (X, g). To obtain global invariants we subtract off contributions from the background metric (X_0, g_0). Since $\alpha_j(g)$ agrees with $\alpha_j(g_0)$ on $X - K \cong X_0 - K_0$, we define the relative heat invariant as

$$a_j(g, g_0) := \int_K \alpha_j(g)\, dg - \int_{K_0} \alpha_j(g_0)\, dg_0. \tag{4.2}$$

By the formula connecting the heat and wave operators,

$$e^{-u(\Delta_g - n^2/4)} = \frac{1}{\sqrt{\pi u}} \int_0^{\infty} e^{-t^2/4u} \cos\left(t\sqrt{\Delta_g - n^2/4}\right)\, dt, \tag{4.3}$$

and the characterization of the wave kernel in Joshi-Sá Barreto [33], we can see that the heat kernel has a well-defined 0-trace (i.e. its kernel is polyhomogeneous in ρ as $\rho \to 0$).

We could try to define regularized heat invariants directly from the 0-trace of the $\alpha_j(g)$'s. For conformally compact Einstein manifolds, Albin [1] shows that that these 0-traces give well-defined invariants. In our situation it is simpler to consider only the expansion of the relative heat trace, and the corresponding relative heat invariants, for which any possible dependence on the regularization scheme is effectively canceled.

Proposition 4.1. *The difference of heat* 0*-traces admits an expansion in terms of relative heat invariants,*

$$0\text{-tr}\big(e^{-t\Delta_g}\big) - 0\text{-tr}\big(e^{-t\Delta_{g_0}}\big) \sim t^{-\frac{n+1}{2}} \sum_{j=0}^{\infty} t^j a_j(g, g_0).$$

Proof. By the local form of the heat expansion (4.1), we see immediately that

$$\int_K H_g(t; z, z)\, dg(z) - \int_K H_{g_0}(t; z, z)\, dg_0(z) \sim t^{-\frac{n+1}{2}} \sum_{j=0}^{\infty} t^j a_j(g, g_0).$$

Hence the goal is to show that

$$\sideset{^0}{}\int_{X_0 - K_0} \big[H_g(t; z, z) - H_{g_0}(t; z, z)\big] dg_0(z) = O(t^{\infty}), \tag{4.4}$$

as $t \to 0$, where we implicitly make use of the isometry $(X - K, g) \cong (X_0 - K - 0, g_0)$ to combine the two 0-integrals.

To estimate (4.4) near infinity we introduce cutoff functions $\psi_1, \psi_2 \in C^{\infty}(X_0 - K_0)$, both zero on ∂K_0 and 1 near infinity, with $\psi_1 = 1$ on some open neighborhood of the support of ψ_2. After pullback by isometry (which we suppress from

the notation), we can regard $\psi_2 e^{t\Delta_g}\psi_1$ as an operator on $L^2(X_0 - K_0, dg_0)$. By integrating

$$\frac{d}{du}\Big[\psi_2 e^{-u\Delta_g}\psi_1 e^{-(t-u)\Delta_{g_0}}\psi_1\Big] = -\psi_2 e^{-u\Delta_g}[\Delta_{g_0}, \psi_1] e^{-(t-u)\Delta_{g_0}}\psi_1,$$

we obtain a cutoff version of Duhamel's formula,

$$\psi_2 e^{-t\Delta_g}\psi_1 - \psi_1 e^{-t\Delta_{g_0}}\psi_2 = -\int_0^t \psi_2 e^{-u\Delta_g}[\Delta_{g_0}, \psi_1] e^{-(t-u)\Delta_{g_0}}\psi_2 \, du.$$

Choose $\eta \in C_0^\infty(X_0 - K_0)$ such that $\eta = 1$ on the support of $[\Delta_{g_0}, \psi_1]$ and so that the supports of η and ψ_2 are separated by distance $\delta > 0$. We can rewrite the above formula as

$$\psi_2 e^{-t\Delta_g}\psi_1 - \psi_1 e^{-t\Delta_{g_0}}\psi_2 = -\int_0^t A_1(u) A_2(t-u)\, du, \tag{4.5}$$

where

$$A_1(u) := \psi_2 e^{-u\Delta_g}\eta,$$

and

$$A_2(u) := [\Delta_{g_0}, \psi_1] e^{-u\Delta_{g_0}}\psi_2.$$

Using the estimates of Cheng-Li-Yau [15, Cor. 8] for the heat kernel on complete manifolds with bounded curvatures, we can estimate the kernels of the $A_i(u)$ by

$$A_i(u; z, w) \le C_i u^{-(n+i)/2} e^{-cd(z,w)^2/u}.$$

Since the kernels are smooth and decay rapidly at infinity, we conclude that the $A_i(u)$'s are Hilbert-Schmidt. Moreover, because the $d(z, w) \ge \delta$ in the supports of the cutoffs, we can estimate the Hilbert-Schmidt norms by

$$\|A_i(u)\|_2 \le C_i e^{-c\delta^2/u}.$$

From (4.5) we can then estimate the trace norm

$$\left\|\psi_2 e^{-t\Delta_g}\psi_1 - \psi_1 e^{-t\Delta_{g_0}}\psi_2\right\|_1 = O(t^\infty).$$

This shows that the 0-integral in (4.4) is a convergent integral and that

$$\int_{X_0 - K_0} \psi_2(z)\big[H_g(t; z, z) - H_{g_0}(t; z, z)\big] dg_0(z) = O(t^\infty),$$

Finally, on $X_0 - K_0$, we have $\alpha_j(g) = \alpha_j(g_0)$, so that the estimate,

$$\int_{X_0 - K_0} (1 - \psi_2(z))\big[H_g(t; z, z) - H_{g_0}(t; z, z)\big] dg_0(z) = O(t^\infty),$$

follows from the local heat expansion (4.1). □

If we assume knowledge of the the relative scattering phase, then it is relatively easy to recover relative heat invariants via the wave trace.

Proposition 4.2. *For (X_0, g_0) conformally compact and hyperbolic near infinity and $(X, g) \in \mathcal{M}(X_0, g_0, K_0)$, the relative scattering phase $\sigma(\xi)$ determines the relative heat invariants $a_j(g, g_0)$.*

Proof. By Proposition 3.1, the relative scattering phase determines the difference of the wave 0-traces for g and g_0. Using the relation (4.3) between the heat and wave operators, we can then apply Proposition 4.1 to recover the relative heat invariants. □

In even dimensions we are able to get more information out of the resonance set, following the methods of [8], with some restrictions on the background metric. We will only make application of these results in dimension two (see §7), but we may as well give the proof for any even dimension.

For this argument, assume that (X, h) is conformally compact hyperbolic and that g is another metric on X that agrees with h to order ρ^2. (This easing of the restriction that g and h agree outside a compact set will actually be required for the arguments based on conformal uniformization in §7.) Let $L^2(X)$ denote the space of square-integrable half-densities, with $\hat{\Delta}_g$ and $\hat{\Delta}_h$ the Laplacians on $L^2(X)$ associated to the respective metrics. We deduce that $e^{-t\hat{\Delta}_g} - e^{-t\hat{\Delta}_h}$ is a trace class operator on $L^2(X)$ from Duhamel's formula,

$$e^{-t\hat{\Delta}_g} - e^{-t\hat{\Delta}_h} = \int_0^t e^{-u\hat{\Delta}_g}(\hat{\Delta}_g - \hat{\Delta}_h)e^{-(t-u)\hat{\Delta}_g}\, du.$$

In this context the relative heat trace expansion is given by

$$\operatorname{tr}\Big[e^{-t\hat{\Delta}_g} - e^{-t\hat{\Delta}_h}\Big] \sim t^{-\frac{n+1}{2}} \sum_{j=0}^{\infty} t^j b_j, \tag{4.6}$$

where

$$b_j := \lim_{\varepsilon\to 0}\left[\int_{\{\rho\geq\varepsilon\}} \alpha_j(g)\, dg - \int_{\{\rho\geq\varepsilon\}} \alpha_j(h)\, dh\right]. \tag{4.7}$$

The parametrix construction from [28] shows that the operator $\hat{R}_g(s)^m - \hat{R}_h(s)^m$ is trace class on $L^2(X)$ for $\operatorname{Re} s > n$ with $m = (n+3)/2$. For $\operatorname{Re} w \geq m$ and $\operatorname{Re} s > n$ define the relative zeta function

$$\zeta(w, s) := \operatorname{tr}\big[\hat{R}_g(s)^w - \hat{R}_h(s)^w\big].$$

In terms of heat operators, we have

$$\zeta(w, s) = \frac{1}{\Gamma(w)} \int_0^\infty t^w e^{ts(n-s)} \operatorname{tr}\big[e^{-t\hat{\Delta}_g} - e^{-t\hat{\Delta}_h}\big]\frac{dt}{t}. \tag{4.8}$$

The heat expansions as $t \to 0$ can be used to show that $\zeta(w, s)$ extends meromorphically to $\operatorname{Re} w > -1$, with simple poles at $w = \frac{n+1}{2}, \frac{n-1}{2}, \dots$, ending at 1 for n odd and continuing to negative half-integers for n even. In any dimension $\zeta(w, s)$ is analytic at $w = 0$, and so the relative determinant,

$$D_{\mathrm{rel}}(s) := \exp\big[-\partial_w \zeta(w, s)|_{w=0}\big],$$

is well-defined for $\operatorname{Re} s > n$.

Let $Z_h(s)$ denote the Selberg zeta function for (X, h). Patterson-Perry [39, Thm. 1.9] proved the factorization formula

$$Z_h(s) = e^{p_1(s)} G_\infty(s)^{-\chi(X)} P_h(s), \tag{4.9}$$

where $p_1(s)$ is a polynomial of degree at most $n + 1$ and

$$G_\infty(s) = s \prod_{k=1}^{\infty} E(-\tfrac{s}{k}, n+1)^{h_n(k)},$$

with

$$h_n(k) := (2k+n)\frac{(k+1)\dots(k+n-1)}{n!}.$$

The formula (4.9) remains valid even when $(X,h) = \mathbb{H}^{n+1}$; in this case $Z_h(s) := 1$, and the poles of $G_\infty(s)^{-1}$ cancel the zeroes of $P_h(s)$.

From the proof of [7, Prop 7.2] we see that

$$D_{\mathrm{rel}}(s) := e^{p_2(s)} \frac{P_g(s)}{P_h(s)},$$

with $p_2(s)$ also a polynomial of degree at most $n+1$. Thus we have

$$D_{\mathrm{rel}}(s) := \frac{e^{p(s)} P_g(s)}{Z_h(s) G_\infty(s)^{\chi(X)}}, \tag{4.10}$$

for $p(s)$ a polynomial of degree at most $n+1$.

Proposition 4.3. *Suppose that (X,h) is a conformally compact hyperbolic metric with $\dim X$ even, and g is a metric hyperbolic near infinity that agrees with h to order ρ^2. Then the Euler characteristic $\chi(X)$ and the resonance set $\mathcal{R}_g^{\mathrm{sc}}$ together determine the product $D_{\mathrm{rel}}(s) Z_h(s)$ and all of the relative heat invariants b_j defined by (4.7). When $\dim X = 2$, the set $\mathcal{R}_g = \mathcal{R}_g^{\mathrm{sc}}$ alone determines $\chi(X)$, $D_{\mathrm{rel}}(s) Z_h(s)$, and the relative heat invariants.*

Proof. We examine the asymptotic expansion of $\log D_{\mathrm{rel}}(s)$ as $\operatorname{Re} s \to \infty$. By (4.8) and the heat expansion, we have

$$\begin{aligned} \log D_{\mathrm{rel}}(s) \sim \sum_{j=0}^{\frac{n+1}{2}} & c_{n,j} b_j [s(s-n)]^{\frac{n+1}{2}-j} \log[s(s-n)] \\ & + \sum_{j > \frac{n+1}{2}} c_{n,j} b_j [s(s-n)]^{\frac{n+1}{2}-j}, \end{aligned} \tag{4.11}$$

where the $c_{n,j}$'s are nonzero combinatorial constants.

On the other hand, consider the factorization (4.10). The log of $Z_h(s)$ decays exponentially as $\operatorname{Re} s \to \infty$. Thus $\chi(X_0)$ and $\mathcal{R}_g^{\mathrm{sc}}$ together determine the asymptotic expansion of $p(s) + \log D_{\mathrm{rel}}(s)$ as $\operatorname{Re} s \to \infty$, where $p(s)$ is the polynomial appearing in (4.10). Because of the log terms in (4.11), both the heat invariants and the coefficients of $p(s)$ are fixed by this expansion.

The $n = 1$ case of this result was proven in [8, Prop. 5.8]. in this case, the known asymptotics of $\log G_\infty(s)$ and the vanishing of the first relative heat invariant (by Gauss-Bonnet), allow the Euler characteristic also to be determined from $\mathcal{R}_g^{\mathrm{sc}}$. $\square$

The amusing feature of Proposition 4.3 is that no information on $\mathcal{R}_h^{\mathrm{sc}}$ is needed for the result, because of the structure of the Selberg zeta function. In odd dimensions, the corresponding argument breaks down because the asymptotic formula corresponding to (4.11) is

$$\log D_{\mathrm{rel}}(s) \sim \sum_{j=0}^{\infty} c_{n,j} b_j [s(s-n)]^{\frac{n+1}{2}-j},$$

i.e. there are no logarithmic terms. The absence of such terms means we cannot rule out cancelation between the coefficients of $p(s)$ and the relative heat invariants $b_0, \ldots, b_{n/2}$.

5. Finiteness of topological types

For compact manifolds dimensions greater than 3, the heat invariants do not contain enough information to establish C^k bounds on the curvatures. This problem of course persists in the non-compact case. However, we can certainly use spectral information to control the topological type, following arguments of [14]. The crucial result is the following:

Theorem 5.1 (Grove-Petersen-Wu [23], Thm. C)**.** *The class of closed Riemannian m-manifolds M with injectivity radius bounded below and volume bounded above contains at most finitely many homeomorphism types if $m \geq 4$, and only finitely many diffeomorphism types if $m \geq 5$.*

Fix an asymptotically hyperbolic manifold (X_0, g_0) with boundary defining function ρ and a compact subset $K_0 \subset X_0$. Let $\mathcal{M}(X_0, g_0, K_0)$ denote the class of manifolds X, g such that $(X - K, g) \cong (X_0 - K_0, g_0)$ for some compact $K \subset X$. We will assume that 0-volumes for elements of $\mathcal{M}(X_0, g_0, K_0)$ are defined by boundary defining functions that agree with ρ on $X - K$.

Corollary 5.2. *The set of manifolds in $\mathcal{M}(X_0, g_0, K_0)$ with injectivity radius bounded below and* $\mathrm{vol}(K, g)$ *bounded above contains at most finitely many homeomorphism types if* $\dim X_0 \geq 4$*, and only finitely many diffeomorphism types if* $\dim X_0 \geq 5$.

Proof. Suppose that we glue two copies of K_0 together along a neck N_0, diffeomorphic to $\partial_\infty X \times [-1, 1]$, to form a compact manifold D_0, with metric $\tilde{g}_0$ defined as a smooth extension of the g_0 metric on each copy of K_0. For some $\delta > 0$ we may assume that a region near the edges of $(N_0, \tilde{g}_0)$, defined by

$$Z_{2\delta} := \Big\{ p \in N_0 : \ d(p, \partial N_0) \leq 2\delta \Big\} \subset N_0,$$

is isomorphic to the corresponding region of (X_0, g_0).

We can use the same neck $(N_0, \tilde{g}_0)$ to form the corresponding double $(D, \tilde{g})$ for any $(X, g) \in \mathcal{M}(X_0, g_0, K_0)$. The volume of this double is controlled by

$$\mathrm{vol}(D, \tilde{g}) \leq 2\,\mathrm{vol}(K, g) + \mathrm{vol}(N_0, \tilde{g}_0), \tag{5.1}$$

which is bounded above by assumption.

As for the injectivity radius, we claim that

$$\mathrm{inj}(D, \tilde{g}) \geq c, \tag{5.2}$$

where c depends only on $\mathrm{inj}(X, g)$, the fixed geometry of $(N_0, \tilde{g}_0)$, and δ. Consider first a point $p \in D - N_0$. If a geodesic loop originating at p lies entirely within $K \cup Z_\delta$ (using either copy of K), then its length is bounded below by $2\,\mathrm{inj}(X, g)$. On the other hand, if a point of the geodesic loop intersects $N_0 - Z_\delta$, then the length of the loop is greater than 2δ. The same reasoning applies to any segment connecting p to a conjugate point, so we conclude that $\mathrm{inj}(p)$ satisfies the bound (5.2) in this case. The argument starting from $p \in N_0 - Z_\delta$ is virtually identical.

This leaves the case of $p \in Z_\delta$. If geodesic loop originating at p has length shorter than δ, then it lies completely within $K \cup Z_{2\delta}$ and this length is bounded below by $2\,\mathrm{inj}(X, g)$. Since $(X_0 - K_0, g_0)$ has negative curvature, there are no conjugate points within $Z_{2\delta}$. Thus if a segment joining p to a conjugate point is shorter than δ, it must lie completely within $K \cup Z_{2\delta}$. The length of this segment is then bounded below by $\mathrm{inj}(X, g)$. This completes the proof of (5.2).

Using (5.1) and (5.2), the result now follows from Theorem 5.1. □

It is now straightforward to combine these results with the spectral results from the preceding sections. Note that fixing (X_0, g_0) fixes the d_k contributions to $\mathcal{R}_g^{\mathrm{sc}}$, so it does not matter in the statement of Theorem 1.3 whether we specify $\mathcal{R}_g$ or $\mathcal{R}_g^{\mathrm{sc}}$ for the even dimensional case.

Proof of Theorem 1.3. In even dimensions, fixing $\mathcal{R}$ controls $\mathrm{vol}(K, g)$ and the injectivity radius by Corollary 2.5. The result then follows immediately from Corollary 5.2.

In odd dimensions, extra information is required because the resonance set does not fix the 0-volume. (This would be impossible, because the 0-volume can be made arbitrarily large through the choice of ρ.) To control the volume we must fix the scattering phase and appeal to Proposition 4.2. Since the zeroth relative heat invariant is $\mathrm{vol}(K, g) - \mathrm{vol}(K_0, g_0)$, this fixes $\mathrm{vol}(K, g)$ for metrics in $\mathcal{M}(X_0, g_0, K_0)$ Because the scattering phase determines $\mathcal{R}_g^{\mathrm{sc}}$ (relative to the fixed background set $\mathcal{R}_{g_0}^{\mathrm{sc}}$), Corollary 2.5 gives control over the injectivity radius. The result thus follows by Corollary 5.2. □

6. Geometric compactness theorems

To prove C^∞ compactness of a particular class of metrics, we seek to apply the following C^∞ version of the Cheeger compactness theorem:

Theorem 6.1 (Kasue [34], Croke [16]). *Let (M_j, g_j) be a sequence of compact Riemannian manifolds with uniform bounds of the form:*

$$\mathrm{vol}(M_j, g_j) \le C, \qquad \mathrm{inj}(M_j, g_j) \ge c, \qquad \sup |\nabla^k \mathrm{Ricc}(g_j)| \le C_k.$$

Then, after passing to a subsequence, there exists a manifold M_∞ with diffeomorphisms $\varphi_j : M_\infty \to M_j$ such that the metrics $\varphi_j^* g_j$ converge in the C^∞ topology on M_∞.

This is a modification of the compactness theorem of Kasue [34], which assumes a uniform bound on the diameters of (M_j, g_j). (The original version is more refined, yielding $C^{k,\alpha}$ compactness based on control of derivatives of the curvature up to order k.) Since the spectral data give control of the volumes of the cores (K, g), it is more convenient for us to switch from diameter to volume. This link is provided by Croke [16, Cor. 15], who proves that for any compact m-dimensional Riemannian manifold (M, g),

$$\mathrm{diam}(M, g) \le \frac{2m^m \Omega_m}{\Omega_{m-1}} \frac{\mathrm{vol}(M, g)}{\mathrm{inj}(M, g)^{m-1}},$$

with Ω_m the volume of S^m.

It is tempting to try to generalize Theorem 6.1 to the case of even-dimensional asymptotically hyperbolic manifolds, by replacing the volume estimate with a bound on the 0-volume. (There's no hope of this in odd dimensions because the 0-volume is not invariantly defined.) But at least for surfaces we can see immediately that this does not work. Consider a pair of pants with boundary geodesics of length ℓ_1, ℓ_2, ℓ_3 and funnels attached to each of these. As $\ell_1 \to \infty$ the sequence clearly diverges, but curvature is constant, injectivity radius remains equal to $\min(\ell_2, \ell_3)$, and the 0-volume is also constant at 2π. The obvious doubling argument that one

might try to extend Theorem 6.1 fails here because the injectivity radius of the doubled surface may approach zero.

6.1. **Isoresonant compactness in dimension two.** The two dimensional application of Theorem 6.1 is based on the following intermediate result:

Proposition 6.2. *Suppose (X, g) is a conformally compact surface hyperbolic near infinity, with $K(g)$ denoting the Gaussian curvature. We have bounds*

$$\operatorname{inj}(X, g) \geq c, \qquad \sup |\nabla_g^k K(g)| \leq C_k,$$

for any $k = 0, 1, 2, \ldots$, where the constants $c > 0$ and $C_k > 0$ depend only on the resonant set $\mathcal{R}_g$.

We will defer the somewhat technical proof of Proposition 6.2 to §7.

Proof of Theorem 1.1. Let $\mathcal{A}$ denote a collection of surfaces as described in the statement of the theorem. By Proposition 4.3, 0-vol(X, g) is constant over $\mathcal{A}$. Hence vol(K, g) is constant as well. If we form the doubles $(D, \tilde{g})$, by gluing two copies of each compact regions (K, g) along a common neck N, then we produce a corresponding class $\tilde{\mathcal{A}}$ of compact surfaces $(D, \tilde{g})$. These metrics share a fixed volume and the C^k curvature bounds from Proposition 6.2 extend directly because the same neck is used for every case. As in the proof of Corollary 5.2, the injectivity radius is bounded below in terms of the lower bound on $\operatorname{inj}(X, g)$ from Proposition 6.2, the width of the neck, and the curvature in the neck.

Starting from a sequence $\{(X, g_k)\} \subset \mathcal{A}$, we form doubles $(D, \tilde{g}_k)$. By Theorem 6.1 we can assume, after passing to a subsequence, that there exist diffeomorphisms $\varphi_k : D \to D$ such that $\{\varphi_k^* \tilde{g}_k\}$ converges in the C^∞ topology on D to some metric $\tilde{g}_\infty$. In order to apply this result to the original sequence, we need to make sure that the diffeomorphisms φ_k converge to the identity on the neck.

Let d_N denote the distance function corresponding to the metric on the neck, which is fixed independently of k. Suppose $p_1, ..., p_n$ are points in N, chosen so that the distance functions $d_N(p_i, \cdot)$ collectively provide good sets of coordinates covering all of N. Since D is compact, by passing to a subsequence of $\{\varphi_k\}$ we can assume that $\varphi_k^{-1}(p_i)$ converges to some point $p_{i,\infty} \in D$ as $k \to \infty$, for each $i = 1, ..., n$. For $q \in N$ we have

$$d_{\varphi_k^* \tilde{g}_k}(\varphi_k^{-1}(p_i), \varphi_k^{-1}(q)) = d_N(p_i, q).$$

Because the metrics $\varphi_k^* \tilde{g}_k \to \tilde{g}_\infty$ and $\varphi_k^{-1}(p_i) \to p_{i,\infty}$, this implies that

$$\lim_{k \to \infty} d_{\tilde{g}_\infty}(p_{i,\infty}, \varphi_k^{-1}(q)) = d_N(p_i, q). \tag{6.1}$$

Since all of the neck metrics are isometric, the functions $d_{\tilde{g}_\infty}(p_{i,\infty}, \cdot)$ also provide good sets of coordinates, we conclude from (6.1) that $\varphi_k^{-1}(q)$ converges to some point q_∞ such that

$$d_{\tilde{g}_\infty}(p_{i,\infty}, q_\infty) = d_N(p_i, q).$$

This argument shows that the restriction of φ_k^{-1} to N converges to a map $\psi : N \to N$ which is just identity map between the respective coordinate systems defined by $\{d_N(p_i, \cdot)\}$ and $\{d_N(p_{i,\infty}, \cdot)\}$.

We can extend ψ to a diffeomorphism $D \to D$ in some arbitrary way and, after replacing φ_k by $\psi \circ \varphi_k$, we can assume that φ_k converges to the identity on N. Then

we obtain a solution to the original problem by restricting the resulting sequence to K. $\square$

6.2. **Isophasal compactness in dimension three.** Our compactness argument is actually somewhat easier for $\dim X = 3$, because the extra hypothesis of negative curvature gives us control over the injectivity radius immediately from Corollary 2.5. Since $\mathrm{vol}(K, g)$ is fixed by the first relative heat invariant, the doubling trick is essentially all that we need to adapt standard arguments from the compact case.

The one point to clear up is that we can produce bounds on the Sobolev constants of the compact doubles $(D, \tilde{g})$, using spectral information from the original spaces (X, g). The results of Brooks-Perry-Petersen [14, §2] do not apply verbatim, because they assume knowledge of the eigenvalue spectrum of $(D, \tilde{g})$. Adapting these arguments to our case is a relatively simple matter; we include the details for the sake of clarity of exposition.

Theorem 6.3. *Let (M, g) be a compact m-dimensional Riemannian manifold, and assume*

$$\mathrm{vol}(M, g) \leq C, \qquad \mathrm{inj}(M, g) \geq c.$$

Then for each p the constant C_p in the Sobolev inequalities: for $f \in C^\infty(M)$

$$\|f\|_{\frac{pm}{m-p}} \leq C_p\big(\|f\|_p + \|\nabla f\|_p\big) \qquad 1 \leq p < m,$$

and

$$\|f\|_\infty \leq C_p\big(\|f\|_p + \|\nabla f\|_p\big) \qquad p > m,$$

is bounded above by a constant that depends only on p, c, and C.

Proof. By [16, Thm. 14], for any $r \leq \frac{1}{2}\,\mathrm{inj}(M, g)$ we have

$$\frac{\mathrm{vol}(\partial B(p; r))^m}{\mathrm{vol}(B(r))^{m-1}} \geq \frac{2^{m-1}\Omega_{m-1}^m}{\Omega_m^{m-1}}, \tag{6.2}$$

where Ω_m is the volume of S^m. Moreover, this bound can be integrated [16, Prop. 15], yielding, for any $r \leq \frac{1}{2}\,\mathrm{inj}(M, g)$,

$$\mathrm{vol}(B(p; r)) \geq \frac{2^{m-1}\Omega_{m-1}^m}{m^m \Omega_m^{m-1}}\, r^m. \tag{6.3}$$

Fix $r = \frac{1}{2}\,\mathrm{inj}(M, g)$. If we pack M with a maximal collection of disjoint balls $B(p_j, r/2)$, $j = 1, \ldots, k$, then (6.3) gives a bound on the number k of such balls:

$$k \leq \frac{2m^m \Omega_m^{m-1}}{\Omega_{m-1}^m} \frac{\mathrm{vol}(M, g)}{r^m}. \tag{6.4}$$

For $f \in C_0^\infty(B(p; r))$, we can now apply [14, Cor. 2.1], which gives the claimed Sobolev bounds in this case with constants controlled by virtue of (6.2) and (6.3). A simple partition of unity argument (see [14, pp. 78–9]) applied to the cover $\{B(p_j, r)\}_{j=1}^k$, together with the bound (6.4), then extends the result to $f \in C^\infty(M)$. $\square$

Proof of Theorem 1.2. Let $\mathcal{A} \subset \mathcal{M}(X_0, K_0, g_0)$ be a collection as in the statement of the theorem. According to Proposition 4.2, fixing the relative scattering phase fixes the relative heat invariants. Since the background metric is held constant, this in turn fixes the integrals

$$a_{j,K}(g) := \int_K \alpha_j(g)\, dg, \tag{6.5}$$

where $\alpha_j(g)$ is the j-th local heat invariant of Δ_g, as in (4.1). In particular, the $j = 0$ case shows that $\mathrm{vol}(K, g)$ is fixed for $(X, g) \in \mathcal{A}$. By the assumption of negative curvature, the injectivity radius of (X, g) is fixed by Corollary 2.5.

Now we form the collection $\tilde{\mathcal{A}}$ of doubles $(D, \tilde{g})$ as in the proof of Corollary 5.2. The volume and injectivity radius of any $(D, \tilde{g}) \in \tilde{\mathcal{A}}$ are controlled just as in that proof. Theorem 6.3 therefore gives uniform control of the Sobolev constants of $(D, \tilde{g})$. And using the constants $a_{j,K}(g)$, together with the corresponding integrals over the fixed neck, we see that the heat invariants of $(D, \tilde{g})$ are fixed for the collection $\tilde{\mathcal{A}}$.

The final step is to apply the bootstrap argument to produce C^k bounds on the Ricci tensor from the heat invariants, using the Sobolev inequalities. For compact manifolds of dimension three this was done in Brooks–Petersen–Perry [14, §5], and we will not repeat the details here. (See also the nice expository account of this argument in Brooks [10].) □

7. Curvature estimates in dimension two

The main issue in two dimensions is to control the injectivity radius without assuming the curvature is negative. The tool for accomplishing this is conformal uniformization, which was also the basis for the results of Osgood-Phillips-Sarnak [38] as well as Borthwick-Judge-Perry [8].

For conformally compact manifolds, the relevant uniformization theorem follows from the work of Mazzeo-Taylor [37]. There results show in particular that any metric $\bar{g}$ on $\bar{X}$ is conformally related to a unique complete hyperbolic metric, with control of the boundary regularity of the conformal factor. By [8, Cor. 4.2], we can assume an extra order of vanishing of the conformal factor when $K(g) = -1 + O(\rho^2)$. In particular we have the following corollary to the Mazzeo-Taylor result:

Proposition 7.1. *If (X, g) is a conformally compact surface hyperbolic near infinity, then there exists a unique $\varphi \in \rho^2 C^\infty(\bar{X})$ such that*

$$g = e^{2\varphi} h,$$

where h is a complete hyperbolic metric on X.

The compactness arguments cited above [8, 38] rely on the production of a convergent subsequence of uniformizing hyperbolic metrics, which allows reduction to the case of a single fixed background metric h. In the non-compact case [8] this approach requires unfortunate extra restrictions: compact support for the φ and upper bounds on the diameters of funnels for the h.

The argument presented in this section differs from the previous approaches (including Osgood-Phillips-Sarnak) in that the background metric h is never fixed. Instead, we rely on uniform control of the resolvent $R_h(s)$ to turn $H^k(X, dh)$ bounds

on φ into C^k bounds on $K(g)$. We can then exploit the fact that $K(g)+1$ is compactly supported and avoid any restriction on the support of φ.

It is quite possible that the approach presented here could be extended to surfaces with cusps. The conformal uniformization results one would need to use have recently been proved by Ji-Mazzeo-Sesum [31] (for finite volume only) and Albin-Aldana-Rochon [2] (for the general case).

Suppose we take $(X,g), h, \varphi$ as in Proposition 7.1 and apply Proposition 4.3 to the pair g,h. This shows that $\chi(X)$ and the relative heat invariants b_j, defined in (4.7), are determined by $\mathcal{R}_g$. The zeroth relative heat invariant is

$$b_0 = \frac{1}{4\pi}\int_X (e^{2\varphi}-1)\,dh. \tag{7.1}$$

Proposition 4.3 also tells us that the product $D_{\mathrm{rel}}(s)Z_h(s)$ is an invariant of $\mathcal{R}_g$. In particular, the invariant quantity

$$d_0 := \log D_{\mathrm{rel}}(1)Z_h(1),$$

will play an important role here. This is because of the Polyakov formula [41, 3], which was extended to the asymptotically hyperbolic context in [8, Prop. 1.2]:

$$\log D_{\mathrm{rel}}(1) = -\frac{1}{6\pi}\int_X \left(\tfrac{1}{2}|\nabla_h\varphi|^2 - \varphi\right) dh. \tag{7.2}$$

We should note that, in contrast to the compact case [38], $\log D_{\mathrm{rel}}(1)$ is not an invariant of $\mathcal{R}_g$. Fortunately, the quantity d_0 makes a suitable replacement.

For this section it will be convenient to use the notation

$$A \preceq B \quad\Longleftrightarrow\quad A \le CB,$$

where $C>0$ depends only on the invariants of $\mathcal{R}_g$, namely d_0 and $b_0, b_1, \dots$. For example, we claim that

$$\log D_{\mathrm{rel}}(1) \succeq 1.$$

To prove this, we note that the product formula for the Selberg zeta function,

$$Z_h(1) := \prod_{\ell\in\mathcal{L}_h}\prod_{k=1}^{\infty}\left[1-e^{-k\ell(\gamma)}\right], \tag{7.3}$$

where $\mathcal{L}_h$ denotes the primitive length spectrum of (X,h), converges in some neighborhood of 1. (The hyperbolic surface (X,h) has infinite area, so the exponent of convergence for the associated Fuchsian group is strictly less than 1.) For $(X,h)\cong\mathbb{H}^2$ we set $Z_h(s) := 1$. In all other cases, the convergence of (7.3) implies that $Z_h(1)\in(0,1)$. Hence we have a lower bound for $\log D_{\mathrm{rel}}(1)$ that depends only on d_0.

Lemma 7.2. *For g,h as given by Proposition 7.1, we have bounds*

$$\left|\int_X \varphi\,dh\right| \preceq 1, \qquad \int_X |\nabla_h\varphi|^2\,dh \preceq 1, \qquad \int_X |\varphi|^2\,dh \preceq 1, \tag{7.4}$$

along with

$$\mathrm{inj}(X,h) \succeq 1, \qquad \inf\sigma(\Delta_h) \succeq 1. \tag{7.5}$$

The constants in these bounds depend only on the invariants b_0 and d_0.

Proof. The first two bounds were obtained in the proof of [8, Thm. 1.4], but we recall the details for the convenience of the reader. For $\varepsilon > 0$ set $X_\varepsilon := \{\rho \geq 0\} \subset X$ and

$$V_\varepsilon := \mathrm{vol}(\{X_\varepsilon, h)$$

Applying Jensen's inequality with the convex function $F(x) = e^{2x} - 1$ and the probability measure $V_\varepsilon^{-1}\, dh$ on X_ε gives

$$\begin{aligned}\int_{X_\varepsilon} \varphi\, dh &\leq \frac{V_\varepsilon}{2} \log\left[1 + V_\varepsilon^{-1} \int_{X_\varepsilon} (e^{2\varphi} - 1)\, dh\right] \\ &\leq \frac{1}{2} \int_{X_\varepsilon} (e^{2\varphi} - 1)\, dh,\end{aligned}$$

where in the second line we just use $\log(1 + x) \leq x$. Taking $\varepsilon \to 0$ and comparing to (7.1) gives

$$\int_X \varphi\, dh \leq 2\pi b_0. \tag{7.6}$$

From (7.2) and the fact that $\log D_{\mathrm{rel}}(1) \geq d_0$ we then deduce

$$\begin{aligned}\frac{1}{6\pi} \int_X \varphi\, dh &= \frac{1}{12\pi} \int_X |\nabla_h \varphi|^2\, dh + \log D_{\mathrm{rel}}(1) \\ &\geq d_0.\end{aligned} \tag{7.7}$$

Together, (7.6) and (7.7) give the first bound in (7.4).

We can now use the first bound to eliminate the φ term from the Polyakov formula (7.2). This yields the second bound, in the form

$$\int_X |\nabla_h \varphi|^2\, dh \leq 4\pi b_0 - 12\pi d_0,$$

as well as the useful estimate

$$-\log Z_h(1) \leq \frac{b_0}{3} - d_0. \tag{7.8}$$

If $\ell_0(h) := \inf \mathcal{L}_h$ then by (7.3) we have

$$Z_h(1) \leq 1 - e^{-\ell_0(h)},$$

and so (7.8) gives a lower bound

$$\mathrm{inj}(X, h) = \frac{\ell_0(h)}{2} \succeq 1. \tag{7.9}$$

For the remainder of the argument, we apply a result of Dodziuk et al. [18, Thm. 1.1′], which allows one to estimate small eigenvalues of an infinite-area hyperbolic surface in terms of lengths of chains of disjoint simple closed geodesics. In its simplest form, this result implies

$$\inf \sigma(\Delta_h) \succeq \ell_0(h), \tag{7.10}$$

where the constant depends only on the topology of X. This gives the second half of (7.5). The third bound in (7.4) now follows from second bound and the Poincaré inequality,

$$\int_X |\varphi|^2\, dh \leq \frac{1}{\inf \sigma(\Delta_h)} \int_X |\nabla_h \varphi|^2\, dh. \tag{7.11}$$

□

One very useful consequence of the lower bound on the bottom of the spectrum of Δ_h is that it gives uniform control of the heat-kernel $H_h(t,z,w)$ of Δ_h. The results of Davies-Mandouvalos [17, Thm. 5.4] yield the following estimate:

$$H_h(t,z,w) \le C_0 t^{-1} e^{-at} e^{-d(x,w)^2/Dt}, \tag{7.12}$$

for any $0 < a \le \inf \sigma(\Delta_h)$ and $D > 4$. The constant C_0 depends only on the choice of a and D. Lemma 7.2 thus shows that (7.12) holds with constants that depend only on b_0 and d_0.

At this point we've gotten all the information we can out of b_0. And $b_1 = 0$ because $a_1(g) = a_1(h) = -2\pi\chi(X)$. So the next step is to bring in the second relative heat invariant,

$$b_2 = \frac{1}{60\pi} \int_X \left[e^{-2\varphi}(\Delta_h\varphi - 1)^2 - 1 \right] dh. \tag{7.13}$$

Lemma 7.3. *For φ as in Proposition 7.1,*

$$\sup_X |\varphi| \preceq 1,$$

where the constant depends only on the invariants b_0, b_2, and d_0.

Proof. To handle b_2, we need a Trudinger-type inequality with suitable control of the constants. By a theorem of Grigor′yan [22], the Davies–Mandouvalos bound (7.12) implies the Faber-Krahn inequality:

$$\lambda_1(\Omega) \succeq \mathrm{vol}(\Omega)^{-1},$$

for any precompact region $\Omega \subset X$. This allows us to apply some very general results on Sobolev inequalities due to Bakry et al. [5]. In particular, by [5, Thm. 10.1] the Faber-Krahn inequality is equivalent to a family of bounds:

$$\|u\|_r^r \le (C\|\nabla_h u\|_2)^{r-s}\, \|u\|_s^s, \tag{7.14}$$

for any $0 < s < r < \infty$, where $\|\cdot\|_p$ refers to $L^p(X, dh)$. The constant C depends only on the Faber-Krahn constant, which in turn depends only on b_0 and d_0. Setting $s = 2$ and summing over the cases $r = 2, 3\ldots$ leads immediately to a Trudinger inequality [5, Thm. 3.4],

$$\int_X \exp_2(u)\, dh \le \frac{\|u\|_2^2}{(C\|\nabla_h u\|_2)^2}\ \exp_2(C\|\nabla_h u\|_2), \tag{7.15}$$

where $\exp_2(x) := e^x - 1 - x$.

With the Trudinger inequality we can use b_2 to control the L^2 norm of $e^{-\varphi}\Delta_h\varphi$. The expansion the of the formula (7.13) for b_2 gives

$$\left\|e^{-\varphi}\Delta_h\varphi\right\|_2^2 \le 60\pi b_2 + \left| \int_X \left(e^{-2\varphi} - 1\right) dh \right| + 2\left| \int_X e^{-2\varphi}\Delta_h\varphi\, dh \right|. \tag{7.16}$$

Here the the second term on the right-hand side may be controlled using (7.15) and Lemma 7.2,

$$\begin{aligned}
\left| \int_X \left(e^{-2\varphi} - 1\right) dh \right| &\le \left| \int_X (-2\varphi)\, dh \right| + \int_X \exp_2(-2\varphi)\, dh \\
&\preceq \left| \int_X \varphi\, dh \right| + \|\varphi\|_2^2 \\
&\preceq 1.
\end{aligned}$$

The third term of (7.16) is handled similarly, starting from

$$\begin{aligned}\left|\int_X e^{-2\varphi}\Delta_h\varphi\,dh\right| &= \left|\int_X (e^{-2\varphi}-1)\Delta_h\varphi\,dh\right| \\ &\le \left\|e^{\varphi}-e^{-\varphi}\right\|_2 \left\|e^{-\varphi}\Delta_h\varphi\right\|_2.\end{aligned}$$

Since

$$\begin{aligned}\left\|e^{\varphi}-e^{-\varphi}\right\|_2^2 &= \int_X \left[e^{2\varphi}-2+e^{-2\varphi}\right] dh \\ &= \int_X \left[\exp_2(2\varphi)+\exp_2(-2\varphi)\right] dh,\end{aligned}$$

this term can also be bounded by means of (7.15) and Lemma 7.2. Thus from (7.16) we obtain

$$\left\|e^{-\varphi}\Delta_h\varphi\right\|_2^2 \preceq 1 + \left\|e^{-\varphi}\Delta_h\varphi\right\|_2,$$

and we immediately deduce that

$$\left\|e^{-\varphi}\Delta_h\varphi\right\|_2 \preceq 1. \tag{7.17}$$

The next step is to produce an L^p estimate on $R_h(s;z,\cdot)$. For $\operatorname{Re} s > \frac{1}{2}$ we can estimate $R(s;z,w)$ using the heat kernel estimate (7.12) in the formula

$$R_h(s;z,w) = \int_0^\infty e^{s(1-s)t} H_h(t;z,w)\,dt. \tag{7.18}$$

For convenience we set $s = 2$ (although any $s > 1$ would suffice for our argument). For $r := d_h(z,w) \ge 3$, we make the following estimate of (7.18) in terms of the constants C_0, a, D appearing in (7.12):

$$\begin{aligned}R_h(2;z,w) &\le C_0 \int_0^{r/3} t^{-1}e^{-(2+a)t}e^{-r^2/Dt}\,dt + C_0\int_{r/3}^\infty t^{-1}e^{-(2+a)t}e^{-r^2/Dt}\,dt \\ &\le C_0\int_{3r}^\infty e^{-u/D}\,du + C_0\int_{r/3}^\infty e^{-(2+a)t}\,dt \\ &\le C_0 e^{-3r/D} + C_0 e^{-(2+a)r/3},\end{aligned}$$

where we substituted $u = r^2/t$ in the second line. Assuming, as we may, that $D \le 9/2$, this yields a uniform bound for $r \ge 3$,

$$R_h(2;z,w) \le 2C_0 e^{-2r/3}.$$

For $r \le 3$, we can split up the integral (7.18) for $R_h(2;z,w)$ to obtain

$$\begin{aligned}R_h(2;z,w) &\le C_0\int_0^{r^2} t^{-1}e^{-r^2/Dt}\,dt + C_0\int_{r^2}^9 t^{-1}\,dt + C_0\int_9^\infty e^{-at}\,dt \\ &\le C_1 - C_2\log r,\end{aligned}$$

where C_1 and C_2 depend only on C_0 and D. The point of keeping track of the constants in these calculations is to obtain estimates solely in terms of $r = d_h(z,w)$ and constants that depend on b_0 and d_0 but are otherwise independent of the uniformizing hyperbolic metric h.

To control the L_p norms uniformly in z, we lift $R_h(z,w)$ to $\mathbb{H}$ and let $\mathcal{F}$ be a fundamental domain corresponding to (X,h). Then to eliminate the z-dependence

we enlarge the domain from $\mathcal{F}$ to $\mathbb{H}^2$ and switch to geodesic polar coordinates centered at z:

$$\begin{aligned}\|R_h(2;z,\cdot)\|_p^p &= \int_{\mathcal{F}} \big|R_h(2;z,w)\big|^p\,dh \\ &\le \int_{\mathbb{H}^2} \big|R_h(2;z,w)\big|^p\,dh \\ &\le 2\pi \int_0^3 \Big[C_1 - C_2 \log r\Big]^p\ \sinh r\,dr \\ &\quad + 2\pi \int_3^\infty (2C_0)^p e^{-2pr/3}\ \sinh r\,dr\end{aligned}$$

The integrals are convergent for $p \ge 2$, so this establishes uniform estimates

$$\|R_h(2;z,\cdot)\|_p \preceq 1, \qquad \text{for } p \ge 2, \tag{7.19}$$

where for each p the constant depends only on b_0 and d_0.

We can now combine the estimates (7.17) and (7.19) to control φ pointwise, starting from

$$\varphi(z) = \int_X R_h(2;z,w)(\Delta_h + 2)\varphi(w)\,dh.$$

This leads immediately to

$$|\varphi(z)| \le \big\|R_h(2;z,\cdot)e^{\varphi}\big\|_2\, \big\|e^{-\varphi}(\Delta_h+2)\varphi\big\|_2. \tag{7.20}$$

To bound the first term in (7.20), we use

$$\begin{aligned}\big\|R_h(2;z,\cdot)e^{\varphi}\big\|_2 &\le \big\|R_h(2;z,\cdot)\big\|_2 + \big\|R_h(2;z,\cdot)(e^{\varphi}-1)\big\|_2 \\ &\le \big\|R_h(2;z,\cdot)\big\|_2 + \big\|R_h(2;z,\cdot)\big\|_4\, \big\|e^{\varphi}-1\big\|_4.\end{aligned}$$

By (7.19) and (7.15), the norms on the right are all bounded by constants that depend only on b_0 and d_0. For the second term in (7.20), we have

$$\begin{aligned}\big\|e^{-\varphi}(\Delta_h+2)\varphi\big\|_2 &\le \big\|e^{-\varphi}\Delta_h\varphi\big\|_2 + 2\big\|e^{-\varphi}\varphi\big\|_2 \\ &\le \big\|e^{-\varphi}\Delta_h\varphi\big\|_2 + 2\big\|\varphi\big\|_2 + 2\big\|(e^{-\varphi}-1)\varphi\big\|_2 \\ &\le \big\|e^{-\varphi}\Delta_h\varphi\big\|_2 + 2\big\|\varphi\big\|_2 + 2\big\|e^{-\varphi}-1\big\|_4\, \|\varphi\|_4.\end{aligned}$$

The first term is bounded by (7.17), and $\|\varphi\|_p$ is covered for $p \ge 2$ by Lemma 7.2 together with (7.14). It is also easy to bound $\|e^{-\varphi}-1\|_4$ by means of (7.15) and Lemma 7.2, since

$$(e^{-\varphi}-1)^4 = \exp_2(-4\varphi) - 4\exp_2(-3\varphi) + 6\exp_2(-2\varphi) - 4\exp_2(-\varphi).$$

Hence, the terms on the right side of (7.20) are bounded by constants that depend only on b_0, b_2, and d_0, and the result is proved. □

With control of the conformal factor $e^{2\varphi}$, we are able to control the lengths of geodesics in (X,g):

Corollary 7.4. *Suppose (X,g) is a conformally compact surface hyperbolic near infinity, and let $\ell_0(g)$ denote the length of the shortest closed geodesic. Then we have*

$$\ell_0(g) \succeq 1,$$

with a constant that depends only on b_0, b_2, and d_0.

Proof. Suppose η is a closed geodesic on (X, g). By Lemma 7.3, we can estimate the g-length by

$$\ell(\eta; g) \succeq \ell(\eta; h).$$

Although η will not be a h-geodesic in general, we still have the bound $\ell(\eta; h) \geq \ell_0(h)$. Since $\ell_0(h)$ is bounded below in terms of d_0, this gives a lower bound on $\ell(\eta; g)$ that depends only on b_0, b_2 and d_0. □

Proof of Proposition 6.2. Since $K(g)$ is not integrable on (X, g), for the sake of estimates it is convenient to replace it by the compactly supported function

$$\Psi := K(g) + 1 = e^{-2\varphi} \Delta_h \varphi.$$

To control $\|K(g)\|_\infty$, we seek to estimate $\|\Delta_h \Psi\|_2$ and then remove the Laplacian using $R_h(2)$ as in the proof of Lemma 7.2.

The third local heat invariant has the form

$$\alpha_3(g) = c_1 |\nabla_g K(g)|^2 + c_2 K(g)^3,$$

where $c_1 \neq 0$ according to [38, Appendix]. Thus the third relative invariant is

$$b_3 = c_1 \int_X |\nabla_g K(g)|^2 \, dg + c_2 \int_X (K(g)^3 e^{2\varphi} + 1) \, dh \tag{7.21}$$

By $g = e^{2\varphi} h$ we have

$$\int_X |\nabla_g K(g)|^2 \, dg = \int_X |\nabla_h K(g)|^2 \, dh = \|\nabla_h \Psi\|_2^2.$$

Noting that

$$\int_X \Psi e^{2\varphi} \, dh = \int_X \Delta_h \varphi \, dh = 0,$$

the second term in b_3 can be reduced to

$$\int_X (K(g)^3 e^{2\varphi} + 1) \, dh = \int_X (\Psi^3 - 3\Psi^2) e^{2\varphi} \, dh - 4\pi b_0.$$

Lemma 7.3 gives us control of $\sup |e^{2\varphi}|$, and the combination of Lemma 7.3 and (7.17) gives a bound on $\|\Psi\|_2$. Thus from (7.21) we obtain

$$\|\nabla_h \Psi\|_2^2 \preceq 1 + \|\Psi\|_3 \tag{7.22}$$

where the constants depend only on b_0, b_2, b_3, and d_0. Using the Solobev inequalities (7.14) we estimate

$$\|\Psi\|_3^3 \preceq \|\nabla_h \Psi\|_2 \, \|\Psi\|_2^2.$$

In conjunction with (7.22), this implies

$$\|\nabla_h \Psi\|_2 \preceq 1. \tag{7.23}$$

Note also that by means of (7.14), we also have an L^p bound

$$\|\Psi\|_p \preceq 1, \tag{7.24}$$

for any $p \geq 2$.

At this point the usual bootstrap approach applies; we sketch the details for the sake of completeness. Assume that from $b_0, b_2, \ldots, b_k, d_0$ we have extracted the bound

$$\left\|\nabla_h^{j-2} \Psi\right\|_2 \preceq 1, \qquad \text{for } j = 2, \ldots, k \tag{7.25}$$

for $k \geq 3$. (We start the induction at $k = 3$ by (7.23) and (7.24).) Note that at this stage we also have

$$\left\|\nabla_g^j \varphi\right\|_2 \preceq 1, \qquad \text{for } j = 0, \dots, k. \tag{7.26}$$

(The φ estimates stay two derivatives ahead of those for Ψ.) According to [38, Appendix], the heat coefficient b_{k+1} takes the form

$$b_{k+1} = c_1 \int_X |\nabla_g^{k-1} K(g)|^2 \, dg + c_2 \int_X K(g)\, |\nabla_g^{k-2} K(g)|^2 \, dg + \text{lower order}, \tag{7.27}$$

with $c_1 \neq 0$, where "lower order" means fewer derivatives of $K(g)$. After replacing $K(g)$ by Ψ, the lower order terms can be estimated directly using the inductive hypothesis (7.25) and some combination of (7.24) and (7.14). We can replace ∇_g by ∇_h using (7.26) to estimate the extra terms generated. Thus from b_{k+1} and the inductive hypothesis we obtain

$$\left\|\nabla_h^{k-1} \Psi\right\|_2^2 \preceq 1 + \int_X |\Psi|\, |\nabla_h^{k-2} \Psi|^2 \, dh. \tag{7.28}$$

Applying the Hölder inequality to the second term gives,

$$\int_X |\Psi|\; \nabla_h^{k-2} \Psi|^2 \, dh \leq \|\Psi\|_2 \left\|\nabla_h^{k-2} \Psi\right\|_4^2$$

Again we turn to (7.14) for the bound

$$\left\|\nabla_h^{k-2} \Psi\right\|_4 \preceq \|\nabla_h^{k-1} \Psi\|_2^{1/2} \, \|\nabla_h^{k-2} \Psi\|_2^{1/2}.$$

By the inductive hypothesis (7.25) we thus derive from (7.28) the estimate

$$\left\|\nabla_h^{k-1} \Psi\right\|_2^2 \preceq 1 + \left\|\nabla_h^{k-1} \Psi\right\|_2,$$

which immediately yields

$$\left\|\nabla_h^{k-1} \Psi\right\|_2 \preceq 1, \tag{7.29}$$

completing the induction.

From the full collection of heat invariants $b_0, b_1, \dots$ we thereby obtain a full set of H^k estimates:

$$\left\|\nabla_h^{k} \Psi\right\|_2 \preceq 1, \qquad \left\|\nabla_h^{k} \varphi\right\|_2 \preceq 1.$$

To extract C^k estimates is now a simple matter. Let P_m be an arbitrary differential operator of order m with coefficients supported in $K \subset X$. From

$$R_h(2)(\Delta_h + 2) P_m \Psi = P_m \Psi,$$

we obtain

$$|P_m \Psi(z)| \leq \|R(s; z, \cdot)\|_2 \left(\|\Delta_h P_m \Psi\|_2 + 2 \|P_m \Psi\|_2 \right) \preceq 1.$$

To complete the proof, we must produce a lower bound on the injectivity radius $\operatorname{inj}(X, g)$. If $K(g) \leq 0$, then $\operatorname{inj}(X, g) = \ell_0(g)/2$ and Corollary 7.4 already supplies the estimate. Otherwise, we have $\kappa := \sup K(g) > 0$ and the C^0 bound derived above gives $\kappa \preceq 1$. In this case the result follows from the standard estimate (see e.g. [40, §6.3.2]),

$$\operatorname{inj}(X, g) \geq \min\left(\frac{\pi}{\sqrt{\kappa}}, \frac{\ell_0(g)}{2} \right).$$

□

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